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ON TWO QUESTIONS CONCERNING TILINGS

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ABSTRACT

We prove that a tiling of the plane by topological disks is locally finite at most boundary points of tiles, confirming a conjecture by Valette. This comes by way of a much more general theorem on tilings of topological vector spaces. We also investigate a question raised by Klee as to whether or not there is a tiling of separable Hilbert space by bounded convex tiles. We present evidence to support the conjecture that the answer is negative.

0. Introduction

A **tiling** of a topological vector space X is a covering of X by sets (called **tiles**) which are the closures of their pairwise-disjoint interiors. This paper is a result of investigations into two questions concerning tilings. The first, raised by Valette in [22], asks whether a tiling of the plane whose tiles are topological disks can have bad behavior at every boundary point of every tile. Specifically, is it possible that every open set which contains a boundary point of a tile intersects infinitely many tiles? Valette conjectured that no such tiling exists; in this paper we will prove his conjecture correct.

The second question which we investigate was raised by Klee in [12]. It asks whether separable (infinite-dimensional) Hilbert space admits any tiling by bounded convex sets. While this difficult question remains for the time unsettled, we will discuss evidence to support the following conjecture: *no infinite-dimensional reflexive separable Banach space admits a tiling by bounded convex sets.*

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The following definitions will be basic to our discussion. For any covering A of a space X we define the set of **frontier points** of A to be the union of the boundaries of the members of A. This set is denoted by the symbol $F(\mathcal{A})$. In the case of a tiling the frontier points are especially interesting. Indeed, since the interiors of the tiles in a tiling $\mathcal{T}$ are pairwise-disjoint, all of the interesting behavior of $\mathcal{T}$ is concentrated at its frontier points. Note that $F(\mathcal{T})$ is closed, since a point interior to a tile is clearly not an accumulation point of $F(\mathcal{T})$. Some important subsets of $F(\mathcal{T})$ are described below.

A **singular point** of a tiling $\mathcal{T}$ is a (frontier) point at which $\mathcal{T}$ is not locally finite (that is, a point every neighborhood of which intersects infinitely many tiles in $\mathcal{T}$). The collection of all singular points of $\mathcal{T}$ is denoted by $S(\mathcal{T})$. Note that, like $F(\mathcal{T})$, the set $S(\mathcal{T})$ is always closed. Note also that Valette's question may now be stated in the following form: is there a tiling $\mathcal{T}$ of $\boldsymbol{R}^2$ by topological disks such that $F(\mathcal{T}) = S(\mathcal{T})$? In Section 1 of this paper we will answer this question by proving that under fairly weak assumptions on the tiling $\mathcal{T}$ and the space X the set $S(\mathcal{T})$ is nowhere dense in the set $F(\mathcal{T})$.

Even if one is concerned only with tilings of finite-dimensional spaces, singular points arise in many interesting and natural examples (see [9], pp. 114–116). For instance, any tiling of $\boldsymbol{R}^d$ by bounded convex sets, at least one of which is not a polyhedron, must have singular points (see Theorem 5.1 of [13]). In infinite-dimensional settings, singular points are often unavoidable. Corson [5] proved that if X is a Banach space with an infinite-dimensional reflexive subspace then there is no locally finite cover of X by bounded convex sets. Thus, in particular, there can be no **tiling** $\mathcal{T}$ of such a space by bounded and closed convex sets such that $S(\mathcal{T}) = \emptyset$. In fact, we will show in Section 2 that if X is a Banach space with an infinite-dimensional separable reflexive closed subspace, and if $\mathcal{T}$ is a tiling of X by convex sets, then any bounded tile in $\mathcal{T}$ contains uncountably many singular points of $\mathcal{T}$. See [1,2,3,4,8,17,18,22,23] for other results relating to singular points of tilings.

One interesting subset of $S(\mathcal{T})$ is $I(\mathcal{T})$, the set of **improper** points of $\mathcal{T}$. We say that $x \in F(\mathcal{T})$ is **improper** if and only if x belongs to only one tile (see [4,8,18]).

A frontier point x of $\mathcal{T}$ is said to be **protected** if and only if it is interior to the union of the tiles that contain it. The set of protected frontier points of $\mathcal{T}$ is denoted by $P(\mathcal{T})$. (We have of course, $P(\mathcal{T}) \cap I(\mathcal{T}) = \emptyset$.) One subset of $P(\mathcal{T})$

is $P_2(\mathcal{T})$, the collection of protected frontier points common to exactly two tiles. We call these points **2-protected**.

Finally, a point $x \in F(\mathcal{T})$ is said to be **resolvable** if and only if for each neighborhood U of x there is a neighborhood $V \subset U$ of x such that $T \cap V$ is connected for each tile $T \in \mathcal{T}$. The set of resolvable frontier points of $\mathcal{T}$ is denoted by $R(\mathcal{T})$.

Throughout this paper we will use the symbols $\mathrm{cl}\,(A)$, $\mathrm{int}\,(A)$, and $\mathrm{bdy}\,(A)$ to denote the closure, interior, and boundary of the set A. If A is a collection of subsets of the space X then $\bigcup \mathcal{A}$ and $\bigcap \mathcal{A}$ will denote, respectively, the union and intersection of the members of A.

1. The behavior of a tiling at a frontier point

In this section we investigate Valette's conjecture and broaden the question to the more general problem "how poorly may a tiling behave at how many of its frontier points?" Perhaps not surprisingly, the answer will depend on what additional assumptions are placed on the tiling. Our finite imaginations allow us only to "picture" tilings which are somewhat well-behaved at most of their frontier points. For example, we imagine most frontier points to be both 2-protected and resolvable. Even in the plane, however, topological realities go far beyond our intuition when no constraints are present. The good behavior we expect is not at all guaranteed, as the following example shows.

1.1 Example: As noted by Grünbaum and Shephard ([9], p. 55), a construction similar to that of the "Lakes of Wada" (see Yoneyama's paper [24], pp. 60–62, as well as papers by Knaster [14] and Kuratowski [15]) can produce a tiling of $\boldsymbol{R}^2$ (by any finite number or an infinity of tiles) such that *every* frontier point is common to *all* tiles. The tiles, of course, are wildly shaped; but topologically they are the closures of sets homeomorphic to the open unit disk. Here, we describe a tiling $\mathcal{T}$ of the real number line by infinitely many tiles such that the boundary of each tile is $F(\mathcal{T})$ (thus $F(\mathcal{T}) = S(\mathcal{T})$). This extends to any space by taking the product of these tiles with a hyperplane.

Let $\mathcal{I}_1$ be the collection of intervals

$$\{[1/3, 2/3] + k \mid k \in \boldsymbol{Z}\},$$

$\mathcal{I}_2$ the collection

$$\{[1/9, 2/9] + k \mid k \in \boldsymbol{Z}\} \cup \{[7/9, 8/9] + k \mid k \in \boldsymbol{Z}\},$$

and in general, let $\mathcal{I}_n$ be the collection of "middle thirds" of the components of

$$\boldsymbol{R} \setminus \left(\bigcup_{i=1}^{n-1} \mathcal{I}_i \right).$$

Now let

$$\begin{aligned} T_1 &= \operatorname{cl}(\mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_4 \cup \mathcal{I}_7 \cup \mathcal{I}_{11} \cup \cdots), \\ T_2 &= \operatorname{cl}(\mathcal{I}_3 \cup \mathcal{I}_5 \cup \mathcal{I}_8 \cup \mathcal{I}_{12} \cup \cdots), \\ T_3 &= \operatorname{cl}(\mathcal{I}_6 \cup \mathcal{I}_9 \cup \mathcal{I}_{13} \cup \cdots), \end{aligned}$$

and so on. With $\mathcal{T} = \{T_1, T_2, T_3, \ldots\}$ it is clear that $F(\mathcal{T})$ is the union of translates of the standard Cantor set and that every frontier point is common to all tiles in $\mathcal{T}$. ∎

The above example shows that it is possible to have bad behavior at all frontier points of a tiling if the shape and arrangement of the tiles is unrestricted. The question thus becomes "what conditions must be established in order to guarantee good behavior at most frontier points of a tiling?"

We will be concerned with tilings that possess properties defined previously in [18]. We give the definitions here for completeness. Two tilings $\mathcal{T}$ and $\mathcal{T}'$ of the same space X are said to be **topologically equivalent** if and only if there is a homeomorphism $h : X \longrightarrow X$ such that if $T \in \mathcal{T}$ then $h(T) \in \mathcal{T}'$ (thus also if $T' \in \mathcal{T}'$ then $h^{-1}(T') \in \mathcal{T}$). A subset A of a topological vector space X is called **a topological body** if and only if there is a homeomorphism $g : X \longrightarrow X$ such that $g(A)$ is a closed convex set with nonempty interior. (Note that we do not require that $g(A)$ be bounded). We say that a tiling $\mathcal{T}$ of X possesses property

$\mathbf{P}_1$ if and only if $\mathcal{T}$ is topologically equivalent to a tiling by convex sets;

$\mathbf{P}_2$ if and only if each tile in $\mathcal{T}$ is a topological body;

$\mathbf{P}_3$ if and only if for each $T \in \mathcal{T}$, $x \in \operatorname{bdy}(T)$, and neighborhood U of x, there is an open neighborhood $V \subset U$ of x such that $V \setminus \operatorname{bdy}(T)$ consists of exactly two connected components;

$\mathbf{P}_4$ if and only if for each proper subcollection $\mathcal{S} \subset \mathcal{T}$ and each pair of distinct tiles T_1 and T_2 in $\mathcal{S}$, the set $T_1 \cap T_2 \cap \operatorname{bdy}(\bigcup \mathcal{S})$ is nowhere dense in $\operatorname{bdy}(\bigcup \mathcal{S})$.

The Schönflies Theorem states that any homeomorphism between a simple closed curve in $\boldsymbol{R}^2$ and the unit circle may be extended to a homeomorphism of $\boldsymbol{R}^2$ onto itself. It follows that any tiling of $\boldsymbol{R}^2$ by topological disks possesses property $\mathbf{P}_2$. In Theorem 2.2 of [18] it is proved that

$$\mathbf{P}_1 \Longrightarrow \mathbf{P}_2 \Longrightarrow \mathbf{P}_3 \Longrightarrow \mathbf{P}_4.$$

We will establish Valette's conjecture below by proving that if X is a suitably nice space and $\mathcal{T}$ is a tiling of X possessing property $\mathbf{P}_3$ then the set $S(\mathcal{T})$ of singular points of $\mathcal{T}$ is nowhere dense in the set $F(\mathcal{T})$ of all frontier points of $\mathcal{T}$ (in fact, almost all frontier points are both resolvable and 2-protected). Note that this improves considerably on Valette's original conjecture.

Before presenting the main theorems, we need a lemma which codifies a technique of Klee and Tricot from [13].

1.2 LEMMA: *Let X be a topological space such that every closed subset of X is a Baire space in the subspace topology. Let $\mathcal{A}$ be a countable closed covering of X and for each subset $R \subset X$ define $M(R;\mathcal{A})$ to be the set of all $x \in R$ for which there is a neighborhood U of x and a member $A \in \mathcal{A}$ with $U \cap R \subset A$. (Thus $M(R;\mathcal{A})$ is the set of all points of R at which R is "locally contained in a single member of A.") Then for each R the set $M(R;\mathcal{A})$ is dense and (relatively) open in R.*

Proof: It follows immediately from the definition of $M(R;\mathcal{A})$ that $M(R;\mathcal{A})$ is open in R. Now suppose (to reach a contradiction) that there is an open set U such that

$$\emptyset \neq U \cap R \subset R \setminus M(R;\mathcal{A}).$$

Then for each open $V \subset U$ and $A \in \mathcal{A}$ with $A \cap V \cap R \neq \emptyset$ we would have

$$V \cap (R \setminus A) \neq \emptyset.$$

Since $R \cap A$ is (relatively) closed in R this means that $R \cap A \cap U$ is nowhere dense in $R \cap U$. Thus for each $A \in \mathcal{A}$, $\mathrm{cl}(R) \cap A \cap U$ is nowhere dense in $\mathrm{cl}(R) \cap U$. But $\mathrm{cl}(R) \cap U$ is a Baire space and

$$\bigcup_{A \in \mathcal{A}} \mathrm{cl}(R) \cap A \cap U \supset \mathrm{cl}(R) \cap U\ .$$

This is a contradiction since $\mathcal{A}$ is countable. We conclude that no such open set U may exist, so $M(R;\mathcal{A})$ is dense in A. ∎

If we let $R = F(\mathcal{A})$ in Lemma we obtain the following fact.

1.3 COROLLARY: *Let X be a topological space every closed subset of which is a Baire space in the subspace topology, and let A be a countable closed covering of X. Then given an open set $U \subset X$ intersecting the set $F(\mathcal{A})$ of frontier points of A, there is an open set $V \subset U$ and a member $A \in \mathcal{A}$ such that*

$$\emptyset \neq V \cap F(\mathcal{A}) \subset A .$$

∎

We now consider a countable tiling $\mathcal{T}$ of a space X every closed subset of which is a Baire space. We will show in Theorems 1.4 and 1.6 that if $\mathcal{T}$ possesses property $\mathbf{P}_3$ then $P_2(\mathcal{T}) \cap R(\mathcal{T})$ is dense in $F(\mathcal{T})$, and that a weaker condition guarantees that $P_2(\mathcal{T})$ is dense in $F(\mathcal{T})$. The proofs are applications of Corollary 1.3.

1.4 THEOREM: *Let X be a topological vector space every closed subset of which is a Baire space in the subspace topology (in particular, X may be any Banach space). Let $\mathcal{T}$ be a countable tiling of X possessing property $\mathbf{P}_3$. Then the set $P_2(\mathcal{T}) \cap R(\mathcal{T})$ of resolvable 2-protected frontier points is dense and (relatively) open in the set $F(\mathcal{T})$ of all frontier points of $\mathcal{T}$.*

Proof: If follows easily from the definition of property $\mathbf{P}_3$ that $P_2(\mathcal{T}) \subset R(\mathcal{T})$, and the relative openness of $P_2(\mathcal{T})$ is immediate from its definition. It remains only to show that $P_2(\mathcal{T}) \cap R(\mathcal{T})$ is dense in $F(\mathcal{T})$. Let U be an open set intersecting $F(\mathcal{T})$. We will show that U contains points of the set $P_2(\mathcal{T}) \cap R(\mathcal{T})$.

By Corollary 1.3 there is an open set $V \subset U$ and a tile T such that

$$\emptyset \neq V \cap F(\mathcal{T}) \subset T .$$

Since $\mathcal{T}$ possesses property $\mathbf{P}_3$ there must be an open set $W \subset V$ such that $W \setminus \operatorname{bdy}(T)$, which is $W \setminus F(\mathcal{T})$, consists of exactly two connected components. Each of these components must lie in the interior of a tile, and it is clearly impossible for both components to be subsets of $\operatorname{int}(T)$. It is now easy to see that each point of $W \cap F(\mathcal{T})$ is a point of $P_2(\mathcal{T}) \cap R(\mathcal{T})$. ∎

Since no 2-protected frontier point is singular, we have

1.5 COROLLARY: *If X and $\mathcal{T}$ are as in Theorem 1.4 then the set $S(\mathcal{T})$ of singular points of $\mathcal{T}$ is nowhere dense in $F(\mathcal{T})$.* ∎

Now every improper frontier point is singular, so if $\mathcal{T}$ and X meet the conditions of Theorem 1.4 then it follows from Corollary 1.5 that $I(\mathcal{T})$ is nowhere dense in $F(\mathcal{T})$. As the next theorem shows, to get the density of $P_2(\mathcal{T})$ (instead of $P_2(\mathcal{T}) \cap R(\mathcal{T})$) in $F(\mathcal{T})$ it is enough to have property $\mathbf{P}_4$ (weaker than $\mathbf{P}_3$) *and* the assumption that $F(\mathcal{T}) \setminus I(\mathcal{T})$ is of second category in $F(\mathcal{T})$ at each point of $F(\mathcal{T})$. It will then follow from Corollary 1.7 that $I(\mathcal{T})$ is actually, as before, nowhere dense in $F(\mathcal{T})$.

1.6 Theorem: *Let X be a topological vector space every closed subset of which is a Baire space in the subspace topology. Let $\mathcal{T}$ be a countable tiling of X possessing property $\mathbf{P}_4$ such that $F(\mathcal{T}) \setminus I(\mathcal{T})$ is of second category in $F(\mathcal{T})$ at each point of $F(\mathcal{T})$. Then $P_2(\mathcal{T})$ is dense and (relatively) open in $F(\mathcal{T})$.*

Proof: As before, we need only show density since the relative openness is immediate from the definition. Let U be an open set in X intersecting $F(\mathcal{T})$. We will show that U contains points of $P_2(\mathcal{T})$. Using Corollary we may find an open set $V \subset U$ and a tile $T_0 \in \mathcal{T}$ such that

$$\emptyset \neq V \cap F(\mathcal{T}) \subset \mathrm{cl}\,(V) \cap F(\mathcal{T}) \subset T_0 \, .$$

Now $\mathrm{cl}\,(V) \cap F(\mathcal{T})$ is closed, and thus is a Baire space by assumption. Furthermore,

$$(\mathrm{cl}\,(V) \cap F(\mathcal{T})) \setminus I(\mathcal{T}) \subset \bigcup_{T \in \mathcal{T} \setminus \{T_0\}} T \cap T_0 \cap \mathrm{cl}\,(V) \, .$$

Since $\mathcal{T}$ is countable and the set on the above left is of second category in $F(\mathcal{T})$, there is a tile $T_1 \in \mathcal{T} \setminus \{T_0\}$ such that $T_0 \cap T_1$ is of second category in $\mathrm{cl}\,(V) \cap F(\mathcal{T})$. But $T_0 \cap T_1$ is closed, so there is a connected open set $W \subset V$ such that

$$\emptyset \neq W \cap F(\mathcal{T}) \subset T_0 \cap T_1 \, .$$

Assume (to reach a contradiction) that W contains points not in $T_0 \cup T_1$. Since W is connected this forces

$$\emptyset \neq W \cap \mathrm{bdy}\,(T_0 \cup T_1) \subset W \cap F(\mathcal{T}) \subset T_0 \cap T_1 \, .$$

This contradicts property $\mathbf{P}_4$ (with $\mathcal{S} = \{T_0, T_1\}$). Thus

$$W \subset T_0 \cup T_1 \, ,$$

so every point of $W \cap F(\mathcal{T})$ is a point of $P_2(\mathcal{T})$, which completes the proof. ∎

1.7 COROLLARY: *If X and $\mathcal{T}$ are as in Theorem 1.6 then $S(\mathcal{T})$ (and thus $I(\mathcal{T})$) is nowhere dense in $F(\mathcal{T})$.* ∎

We mention here one more theorem concerning 2-protected frontier points of a tiling. The tilings described in Example 1.1 are "pathological" in the sense that lines intersect the tiles in unexpected ways. In particular, there are lines which contain no segment of the boundary of a tile and yet still contain uncountably many frontier points. If a tiling $\mathcal{T}$ is tame enough to intersect lines in a not-too-wild manner, it can be shown that $P_2(\mathcal{T})$ is dense and relatively open in $F(\mathcal{T})$. This is the content of the following theorem. The proof is too long to include here, but may be found in [19]. We will say that a line L **sections** a set $A \subset X$ if $L \cap A$ is the closure in L of $L \cap \operatorname{int}(A)$, and that L **sections** the tiling $\mathcal{T}$ of X if L sections both T and $X \setminus \operatorname{int}(T)$ for every tile $T \in \mathcal{T}$.

1.8 THEOREM: *Let X be a separable normed space every closed subspace of which is a Baire space in the subset topology. Let $\mathcal{T}$ be a tiling of X such that any line that sections $\mathcal{T}$ contains at most countably many points of $F(\mathcal{T})$. Then $P_2(\mathcal{T})$ is dense and (relatively) open in $F(\mathcal{T})$ (and so, as before, $S(\mathcal{T})$ is nowhere dense in $F(\mathcal{T})$).* ∎

It is clear that category arguments are the mainstay of the above results. So, while the theorems are more than enough to answer the question posed by Valette, they are restricted to countable tilings of Baire spaces. This author admits ignorance of to what degree similar results might hold for tilings of higher cardinality or spaces of poorer structure. The setting for the next section of this paper is a separable Banach space. Here, topology enforces countability of tilings, so the general results we have thus far developed are applicable.

2. Tilings of Banach spaces by bounded convex sets

In this section we address the question "does any infinite-dimensional reflexive separable Banach space admit a tiling by bounded convex sets?" (Tilings by convex sets will hereafter be called **convex tilings** and tilings by bounded convex sets will hereafter be called **bounded convex tilings.**) We will give some geometric arguments to support the conjecture that the answer is negative. First, though, we discuss why the question is interesting.

Finite-dimensional Hilbert spaces are nothing more than the familiar Euclidean spaces $\boldsymbol{R}^d$ with the ℓ^2 norm. These easily admit bounded convex tilings (for example, by d-cubes). The space $H = \ell^2(\aleph_0)$ presents, in many ways, the best generalization of the properties of these Euclidean spaces to an infinite-dimensional setting. It is complete, separable, and has an inner product. It seems natural, from the standpoint of studying tilings, to ask if the property of admitting nice tilings also extends to H.

The question is made more interesting by what happens at the other end of the cardinality spectrum. In a very clever construction, Klee has proved (see [11] and [12]) that Hilbert spaces on very large cardinals admit bounded convex tilings. Specifically, if n is an infinite cardinal for which $n^{\aleph_0} = n$ then the Hilbert space $\ell^2(n)$ admits such tilings. The intermediate case of separable Hilbert space H is yet unsettled and apparently very difficult!

Known facts and examples concerning convex tilings of Banach spaces are few, but some of what is known is very interesting. A recent paper by Fonf [8] establishes that if the separable Banach space X admits a convex tiling $\mathcal{T}_1$ such that $\mathcal{T}_1$ contains a bounded tile and $I(\mathcal{T}_1) = \emptyset$ then X admits a *bounded* convex tiling $\mathcal{T}_2$ which is locally finite (that is, $S(\mathcal{T}_2) = \emptyset$). As mentioned previously, Corson proved this cannot occur when X has an infinite-dimensional reflexive subspace.

The key to our analysis in this paper is the following theorem of Lindenstrauss and Phelps [16]: *if X is an infinite-dimensional reflexive Banach space and $C \subset X$ is a bounded and closed convex set with nonempty interior, then the set* ext (C) *of extreme points of C is uncountable.* (Recall that an **extreme point** of a convex set C is a point $z \in C$ such that if $z = \lambda x + (1-\lambda)y$ with $0 < \lambda < 1$ then either $x \notin C$ or $y \notin C$.) Extensions of this theorem are given in [6] and [10]. We will show that, while each tile in a hypothesized bounded convex tiling $\mathcal{T}$ of X has uncountably many extreme points, only countably many of these may be protected. Since

$$F(\mathcal{T}) \setminus P(\mathcal{T}) \subset S(\mathcal{T}),$$

this suggests an abundance of singular points; possibly an over-abundance in light of our results from Section 1.

Our principal lemmas are stated in terms of coverings, so we will need to update some definitions. Let $\mathcal{C}$ be a covering of a space X and, for each $x \in X$, let $\mathcal{C}(x)$ denote the collection of sets in $\mathcal{C}$ which contain x. Then, as before, x is

protected if and only if x is interior to $\bigcup \mathcal{C}(x)$, and the set of all such points will be denoted by $P(\mathcal{C})$. There is a stronger notion appropriate for coverings: the point y is **well-protected** if and only if y is interior to the set

$$X \setminus \left(\bigcup (\mathcal{C} \setminus \mathcal{C}(y)) \right).$$

The set of all such points will be denoted by $WP(\mathcal{C})$.

If Y is a flat (translate of a linear subspace) in the space X and $\mathcal{T}$ is a tiling of X, we have a covering (but not necessarily a tiling) $\mathcal{T}_Y$ of Y given by

$$\mathcal{T}_Y = \{T \cap Y \mid T \in \mathcal{T}\}$$

It is easy to see that $P(\mathcal{T}) \cap Y \subset WP(\mathcal{T}_Y)$.

We now give a crucial but simple lemma about protected extreme points in a covering by convex sets.

2.1 LEMMA: *Suppose $\mathcal{C}$ is a convex covering of a topological vector space X and $C_0 \in \mathcal{C}$ is such that $C \cap \operatorname{int}(C_0) = \emptyset$ for each $C \in \mathcal{C} \setminus \{C_0\}$. Then $\bigcap \mathcal{C}(x) = \{x\}$ for each $x \in \operatorname{ext}(C_0) \cap P(\mathcal{C})$.*

Proof: Suppose (to reach a contradiction) that there is some $x \in \operatorname{ext}(C_0) \cap P(\mathcal{C})$ and some $y \neq x$ such that $y \in \bigcap \mathcal{C}(x)$. Since $x \in \operatorname{ext}(C_0)$ there is a point w colinear with x and y such that

$$[w, x[\cap C_0 = \emptyset.$$

Choose $z \in \operatorname{int}(C_0)$. Let T_1 and T_2 denote the relative interiors of the triangular disks whose vertex sets are, respectively, $\{x, z, w\}$ and $\{x, y, z\}$. Clearly

$$T_2 \subset \operatorname{int}(C_0).$$

Now $x \in P(\mathcal{C})$, so T_1 must meet some $C \in \mathcal{C}(x) \setminus \{C_0\}$. By assumption, $y \in C$. But the convexity of C then forces

$$C \cap \operatorname{int}(C_0) \supset C \cap T_2 \neq \emptyset,$$

a contradiction (see Figure 1). his proves the lemma. ∎

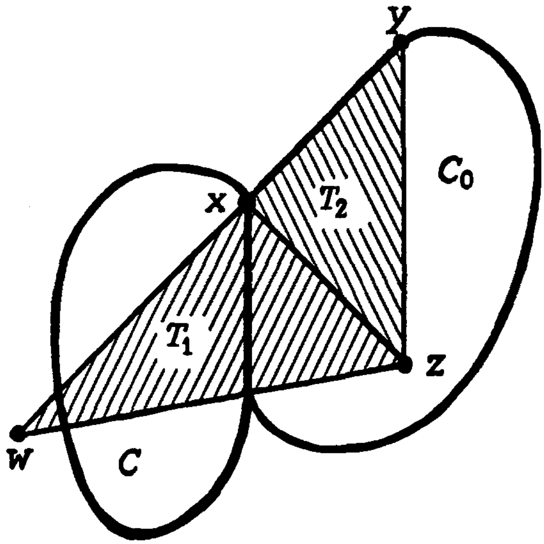

Fig. 1.

The following lemma provides the sought after limitation on the cardinality of protected extreme points.

2.2 LEMMA: *Let $\mathcal{C}$ be a convex convering of a second countable topological vector space X. Let $C_0 \in \mathcal{C}$ be such that $C \cap \operatorname{int}(C_0) = \emptyset$ for each $C \in \mathcal{C} \setminus \{C_0\}$. Then the set $WP(\mathcal{C}) \cap \operatorname{ext}(C_0)$ of well-protected extreme points of C_0 is at most countable.*

Proof: Let $\mathcal{B}$ be a countable base for the topology of X. We will define a one-to-one map

$$WP(\mathcal{C}) \cap \operatorname{ext}(C_0) \longrightarrow \mathcal{B}$$

which will complete the proof.

Consider a point $x \in WP(\mathcal{C}) \cap \operatorname{ext}(C_0)$. Given $y \in (WP(\mathcal{C}) \cap \operatorname{ext}(C_0)) \setminus \{x\}$ there is by Lemma 2.1 some $C_y \in \mathcal{C}(y) \setminus \mathcal{C}(x)$. Thus

$$(WP(\mathcal{C}) \cap \operatorname{ext}(C_0)) \setminus \{x\} \subset \bigcup(\mathcal{C} \setminus \mathcal{C}(x)).$$

Since $x \in WP(\mathcal{C})$ there is a neighborhood $U_x \in \mathcal{B}$ of x with

$$U_x \subset X \setminus \bigcup(\mathcal{C} \setminus \mathcal{C}(x)).$$

Then $U_x \cap (WP(\mathcal{C}) \cap \operatorname{ext}(C_0)) = \{x\}$, so

$$x \longmapsto U_x$$

is the desired map. ∎

As mentioned previously, Corson [5] proved that if X is a Banach space with an infinite-dimensional reflexive subspace then there is no locally finite covering of X by bounded convex sets. The following consequence of Lemma 2.2 may be thought of as a sharpening of Corson's theorem applied to tilings.

2.3 THEOREM: *Let $\mathcal{T}$ be a convex tiling of the Banach space X and let $Y \subset X$ be an infinite-dimensional reflexive separable closed subspace. If $C_0 \in \mathcal{T}$ is bounded and $x \in \operatorname{int}(C_0)$ then the flat $Y + x$ contains uncountably many points of* $\operatorname{bdy}(C_0) \setminus P(\mathcal{T})$.

Proof: By the theorem of Lindenstrauss and Phelps the set $(Y + x) \cap C_0$ has uncountably many extreme points. Lemma 2.2 applies with $X = Y + x$ and $\mathcal{C} = \mathcal{T}_{Y+x}$, so only countably many of these extreme points are well-protected by $\mathcal{T}_{Y+x}$, which implies that only countably many are protected by $\mathcal{T}$. ∎

This theorem implies that if X is a Banach space with an infinite-dimensional reflexive separable subspace and $\mathcal{T}$ is any tiling of X by convex sets, then any bounded tile in $\mathcal{T}$ contains uncountably many unprotected (thus singular) points relative to $\mathcal{T}$. In fact, each point of a bounded tile C_0 is a weak condensation point of $S(\mathcal{T}) \cap C_0$ (every weak neighborhood of a point in C_0 contains uncountably many points of $S(\mathcal{T}) \cap C_0$). The structure of the bounded convex tilings produced by Klee's construction must be very wild indeed!

Similar results for the separable case have been obtained independently by Fonf [7]. Indeed, he has proved that if X is separable and $C_0 \in \mathcal{T}$ is bounded, then $S(\mathcal{T}) \cap C_0$ is weakly dense in C_0 and cannot be covered by countably many weakly closed and nowhere norm-dense subsets.

Theorem 2.3 above gives the following corollary in the separable case.

2.4 COROLLARY: *Let $\mathcal{T}$ be a convex tiling of an infinite-dimensional reflexive separable Banach space X. If $C_0 \in \mathcal{T}$ is bounded and F is any closed infinite-dimensional flat intersecting* $\operatorname{int}(C_0)$ *then* $\operatorname{ext}(F \cap C_0)$ *contains uncountably many points of $S(\mathcal{T})$ (in fact, uncountably many points of $F(\mathcal{T}) \setminus P(\mathcal{T})$).*

Proof: This follows immediately from the above theorem since any closed subspace in X is automatically reflexive (see p. 105 of [20]). ∎

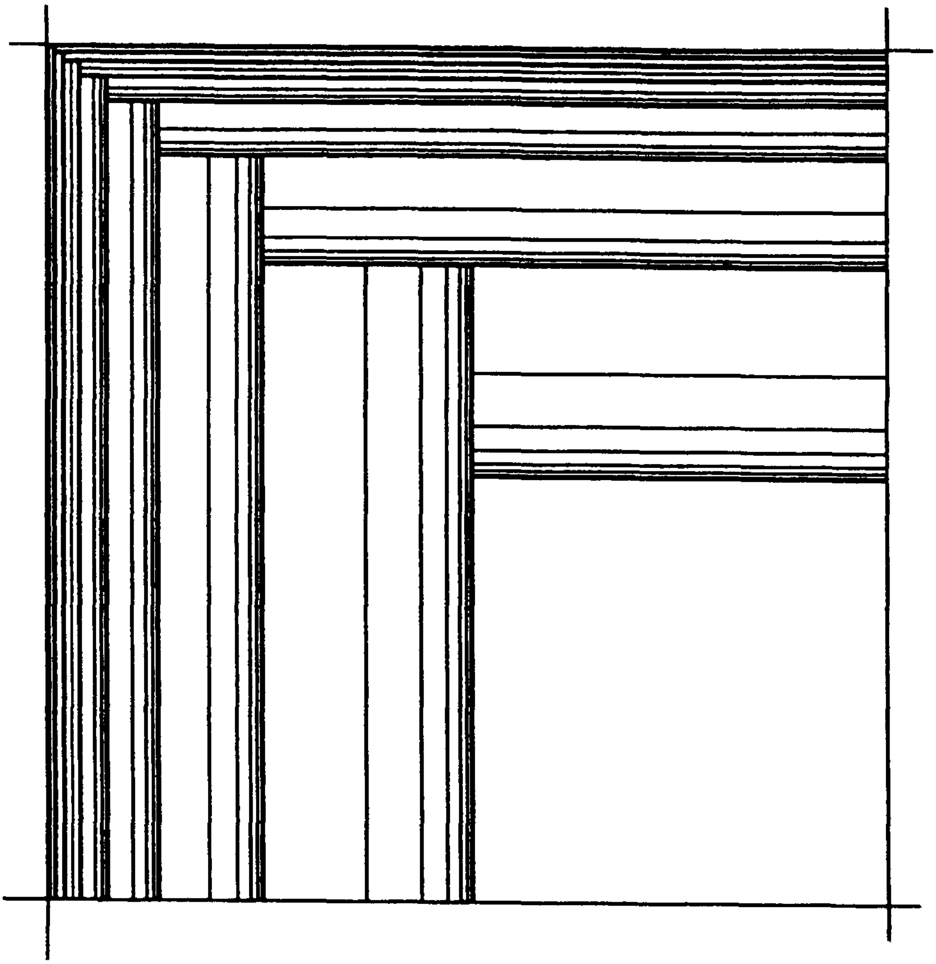

Fig. 2.

In closing we discuss why this suggests an obstacle to the existence of bounded convex tilings of such spaces. Our first impulse might be to suspect that it is impossible for the singular points to be so prevalent in *every* tile. However, we know from Theorem 2.3 applied to Klee's example that such behavior is possible, at least in the nonseparable case. Easier to visualize is the example due to A.H. Stone (see [21]) depicted in Figure 2. Here, the plane is partitioned into squares in the familiar checkerboard arrangement, then each square is subdivided as shown. The result is a bounded convex tiling in which each tile contains uncountably many singular points. In fact, *every* extreme point of *every* tile is singular!

However, our result describes a phenomenon that this two-dimensional example cannot possibly illustrate. If X, $\mathcal{T}$, and C_0 are as stated then the boundary of C_0 is saturated with singular points in the following sense: any glance from inside C_0 which takes in infinitely many dimensions will see uncountably many singular points in bdy(C_0). The author does not know if this will lead to a violation of the conditions derived in Section 1 on the scarcity of singular points, but it seems to point in that direction.

Note that Theorem 2.3 required only one tile to be bounded. In fact, variations on this approach suggest the possibility of the following being true: *if $\mathcal{T}$ is a convex tiling of an infinite-dimensional reflexive separable Banach space then every tile in $\mathcal{T}$ is unbounded.* The author is not aware of any examples to contradict this possibility.

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